HW 2

$\boxed{1}$ $\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \langle X, Y\rangle = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$

$\mathcal{F}\{\text{sinc}(t)\} = \Pi(f)$ — this will be far easier ...

$\mathcal{F}\{\text{sinc}(\frac{t}{B})\} = B\,\Pi(Bf) = \Phi(f)$

$\mathcal{F}\{\phi_n(t)\} = e^{-2\pi j n B f}\,\Phi(f) = B e^{-2\pi j n Bf}\,\Pi(Bf)$

$$\langle \Phi_n, \Phi_m\rangle = B^2 \int_{-\infty}^{\infty} e^{-2\pi j n Bf}\,\Pi(Bf)\, e^{+2\pi j m Bf}\,\Pi(Bf)\, df$$

$$= B^2 \int_{-1/2B}^{1/2B} e^{2\pi j (m-n) f B}\, df$$

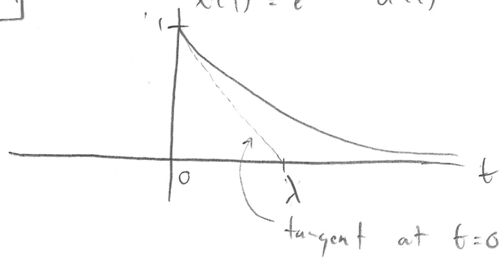
We've done this integral before, just with $\frac{1}{T} = B$.

$$= \boxed{B \cdot \delta[n-m]} = \boxed{\begin{cases} B & \text{if } n = m \\ 0 & \text{else} \end{cases}}$$

In hindsight, I should have picked $\phi(t) = \frac{1}{B}\text{sinc}(\frac{t}{B})$ to get a nice orthonormal set.

$\boxed{2.a}$

$x(t) = e^{-t/\lambda}u(t)$

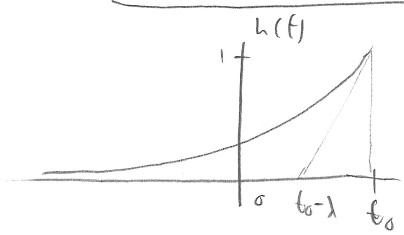



The book draws these tangent lines to give a sense of scale. It's not terribly important in this plot.

$\boxed{2.b}$

We need a matched filter.

$$h(t) = x^*(t_0 - t) = \boxed{e^{-(t_0 - t)/\lambda}\, u(t_0 - t)}$$




$\boxed{2.c}$

The matched filter above is not causal, and thus can't be implemented in a real time system. We want to approximate it with a causal system, but still get 90% of the energy at the peak.

**2.c cont.**

$$y_o = E\{x\} = \int_0^\infty e^{-2t/\lambda}\,dt = \frac{\lambda}{2}(1-0) = \frac{\lambda}{2}$$

$$y_g(t_0) = \int_{-\infty}^{\infty} x(t_0-\tau)\,h(\tau)u(\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-(t_0-\tau)/\lambda}u(t_0-\tau)\,e^{-(t_0-\tau)/\lambda}u(t_0-\tau)u(\tau)\,d\tau$$

$$= \int_0^{t_0} e^{-2(t_0-\tau)/\lambda}\,d\tau = -\int_{t_0}^{0} e^{-2v/\lambda}\,dv = \int_0^{t_0} e^{-2v/\lambda}\,dv = \frac{\lambda}{2}\left(1-e^{-2t_0/\lambda}\right)$$

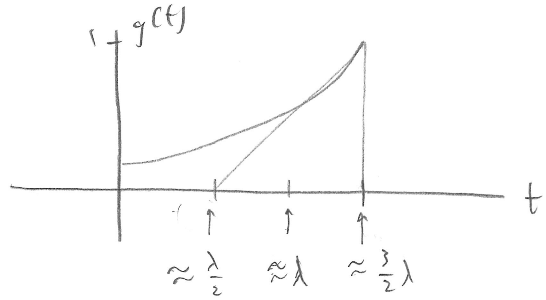
$v = t_0 - \tau$

$$= \alpha\frac{\lambda}{2}$$

$$\alpha = 1 - e^{-2t_0/\lambda} \rightarrow -\frac{2t_0}{\lambda} = \ln(1-\alpha) \Rightarrow t_0 = -\frac{\lambda}{2}\ln(1-\alpha)$$

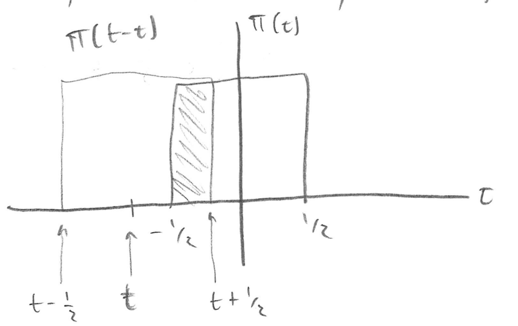
$$\alpha = 0.9 \rightarrow \boxed{t_0 = 1.1513\lambda}$$

clearly, to get $\alpha$ closer to 1, $t_0$ has to get larger.




**3.a**

Probably the easiest way is to just visualize the convolution.




$$x(t) = (\Pi * \Pi)(t)$$

$= 0$ if $t+\frac{1}{2} < -\frac{1}{2} \rightarrow t < -1$

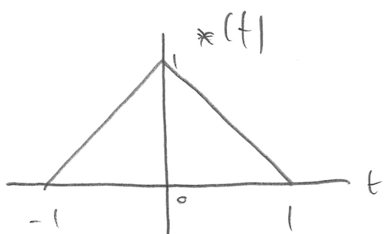
$= 0$ if $t-\frac{1}{2} > \frac{1}{2} \rightarrow t > 1$

$= 1\cdot\left(\left(t+\frac{1}{2}\right)-\left(-\frac{1}{2}\right)\right)$ if $-1 < t < 0$

$= 1+t$

$= 1\cdot\left(\frac{1}{2}-\left(t-\frac{1}{2}\right)\right)$ if $0 < t < 1$

$= 1-t$




$$\boxed{x(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & \text{else} \end{cases}}$$

The triangle pulse is often written as $\Lambda(t)$

3.b

$$\tilde{\mathcal{F}}\{\Lambda(t)\} = \tilde{\mathcal{F}}\{\Pi * \Pi\} = \text{sinc}(f) \cdot \text{sinc}(f) = \boxed{\text{sinc}^2(f)}$$

3.c

$\tilde{\mathcal{F}}\{e^{-t}u(t)\}$ is already in the table. If it wasn't, the integral is simple enough:

$$\int_{-\infty}^{\infty} e^{-t} u(t) e^{-2\pi j f t}\, dt = \int_0^{\infty} e^{-(1+2\pi j f)t}\, dt = -\left.\frac{e^{-(1+2\pi j f)t}}{1+2\pi j f}\right]_{t=0}^{\infty}$$

$$= -\frac{0-1}{1+2\pi j f} = \frac{1}{2\pi j f + 1}$$

Note: $\left|e^{-(1+2\pi j f)t}\right| = e^{-t} \to 0$ as $t \to \infty$

$$e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$$ (time reversal)

$$\mathcal{F}\{e^{-|t|}\} = \boxed{\frac{1}{1+2\pi j f} + \frac{1}{1-2\pi j f}}$$ (frequency reversal)

$$= \frac{1-2\pi j f + 1 + 2\pi j f}{|1+2\pi j f|^2} = \boxed{\frac{2}{1+(2\pi f)^2}}$$

3.d.

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\{\tfrac{1}{2}e^{2\pi j f_0 t} + \tfrac{1}{2}e^{-2\pi j f_0 t}\}$$

$$\mathcal{F}\{\tfrac{1}{2}\} = \tfrac{1}{2}\delta(t), \quad \mathcal{F}\{\tfrac{1}{2}e^{2\pi j f_0 t}\} = \tfrac{1}{2}\delta(f-f_0)$$

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \boxed{\tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0)}$$

## 4.a

$$X'(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-2\pi j \left(\frac{\omega}{2\pi}\right) t} dt = \boxed{X\left(\frac{\omega}{2\pi}\right)}$$

## 4.b

$$\int_{-\infty}^{\infty} X'(\omega) e^{-j\omega t} d\omega = \int_{-\infty}^{\infty} X\left(\frac{\omega}{2\pi}\right) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} X(f) e^{2\pi j f t} 2\pi df = 2\pi x(t)$$

$$f = \frac{\omega}{2\pi} \qquad \boxed{a = \frac{1}{2\pi}}$$

## 4.c

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = b\int_{-\infty}^{\infty} |X'(\omega)|^2 d\omega = b\int_{-\infty}^{\infty} \left|X\left(\frac{\omega}{2\pi}\right)\right|^2 d\omega$$

$$\omega = 2\pi f$$

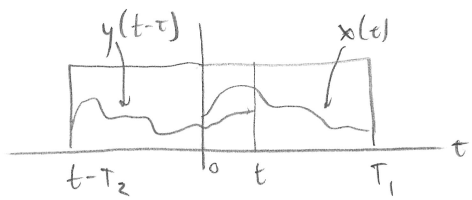
$$= b\int_{-\infty}^{\infty} |X(f)|^2 2\pi df \to b \cdot 2\pi = 1 \to \boxed{b = \frac{1}{2\pi}}$$

## 5.

Convolution is time-invariant, so without loss of generality,

$$x(t) = (u(t) - u(t-T_1))\, x(t) \qquad T_1 = b - a$$
$$y(t) = (u(t) - u(t-T_2))\, y(t) \qquad T_2 = d - c$$

Now it's just like problem 3.a.




0 if $t < 0$

0 if $t - T_2 > T_1 \to t > T_1 + T_2$

possibly non-zero if $t > 0$ or $t > T_1 + T_2$

So the maximum non-zero interval is $\boxed{T_1 + T_2}$